Aligning Normals in a Low-Dimensional Geometric Lens Space

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1 Table of Notations

$gL^n(\star)$: a so called geometric lens space in $n+1$ dimensions
$As^n(\star)$: the aster (and its analogues, such as the hyperbolic octahedron) in $n+1$ dimensions
$Ci^n(\star)$: the circle (and its analogues, such as the 2-sphere) in $n+1$ dimensions
$Sq'^n(\star)$: the square rotated by $\frac{\pi}{4}$ (and its analogues, such as the octahedron) in $n+1$ dimensions
$\triangle$: the Laplacian (defined as the 2nd (partial) derivative)
$\angle(\triangle X)(\triangle Y)$: the angle between to normal vectors of $gL^n(\star)$-components X and Y
$|X-Y|$: the length (distance) between two $gL^n(\star)$-components (via $\triangle$)
Re: a reflection transformation
Ro: a rotation transformation
p_i: a point on a given manifold
p_j: the symmetric twin of a point on a given manifold
e_i: elementary vectors of an $n+1$-dimensional space; exact points where the component functions of $gL^n(\star)$ intersect
$\mathbb{F}^{n+1}$: a field in $n+1$ dimensions
$\star$: any set of variables in a given $n+1$-dimensional space (e.g; Euclidean, elliptical, hyperbolic)

2 Introduction

This brief paper continues the ongoing process of developing and exploring properties of the so-called geometric lens space (henceforth referred to by the symbol $gL^n(\star)$). In this particular paper, as the title indicates, we will focus on the process of aligning normals of $As^n(\star)$ and $Sq'^n(\star)$ in 2-dimensional and 3-dimensional spaces.
The idea of aligning normals was mentioned, but never developed, in previous papers (most specifically in [1] and [2]). This paper seeks explicitly state the process of aligning the normals in the interior of $gL^n(\star)$.
Section 1 will give a very brief background concerning $gL^n(\star)$, though the reader is encouraged to read [1], [2],[3], and [4] and their given erratas (as well as erratas given in [5] and [6]) for a more comprehensive background on the subject. As stated before, this paper is most closely associated with [1], and the background given in Section 1 will reflect that.
Section 2 will give the process required to align the normals of $As^n(\star)$ and $Sq'^n(\star)$ in 2-dimensional and 3-dimensional spaces such that the angle between their respective normals is 0. This process will also explain why the maximum distance between $As^n(\star)$ and $Ci^n(\star)$ (via their normals) is listed as $\frac{\sqrt{2}-1}{\sqrt{2}}$.
The author acknowledges that much of the notation in this ongoing project is non-standard. Thus, a table of notations is given on the page prior to this introduction to aid in clarification.
The author also acknowledges that [1] is rife with typos and errors, as well as the usual mistakes that occur when a lower-division undergraduate attempts to undertake their first research project. . Most typos and errors have been addressed in the errata for [1], as well as in [5] and [6], but some errors are still being discovered (e.g.; on page 12 of [1], it is stated that $\|Ci^n(\star) - Sq'^n(\star)\|$ and $\|Ci^n(\star) - Sq'n(\star)\|$ are not equal when it should be stated that $\|Ci^n(\star) - Sq'^n(\star)\|$ and $\|As^n(\star) - Sq'^n(\star)\|$ are not equal).

3 Section 1: Brief Background

What we shall refer to as $gL^n(\star)$ is a space created from the union of 3 homeomorphic n-manifolds in $n+1$ dimensions for $n \geq 1$: $As^n(\star)$ (a hyperbolic manifold), $Sq'^n(\star)$ (an Euclidean manifold), and $Ci^n(\star)$ (an elliptic manifold). In this paper we deal only with the $n = 1$ and $n = 2$ cases.
$gL(\star)$ can exist in both open and closed forms; the open form occurring when we choose $gL(\star) - \{e_i\}$, which is the elimination of the intersecting points of $As^n(\star)$, $Sq'^n(\star)$, and $Ci^n(\star)$ in their unit representations. In this paper we focus solely on the closed form of $gL(\star)$, but note that all results hold in open form.
When $n = 1$, we use line segments of component manifolds to compose $gL^1(\star)$,

and when $n = 2$, we use triangulations of component manifolds to compose $gL^2(\star)$.
In the $n = 1$ case, $gL^n(\star)$ is completely, simply connected via the union of component segments. In the $n = 2$ case, $gL(\star)$ cannot be considered simply connected since its component parts only intersect at the vertices of their respective triangulations.
(Note:Throughout the remainder of this paper, we will remove the superscript n, with the knowledge that $\{n\} = \{1, 2\}$ for these processes in their respective dimensions.)

4 Section 2: Aligning Normals in $gL^n(\star)$ in $\mathbb{F}^2$ and $\mathbb{F}^3$

In this section we describe the process involved in aligning the normals of $As(\star)$ and $Sq'(\star)$ such that $\angle(\triangle As^n(\star)), (\triangle Sq'^n(\star)) = 0$ for $\{n\} = \{1, 2\}$, meaning every normal vector of $As(\star)$ under transformation is parallel to any normal vector of $Sq'(\star)$, creating a straight, non-intersecting line which we can extend from $As(\star)$ to $Ci(\star)$.
The process for aligning the normals of $gL(\star)$'s component functions depends on the dimension we are working in. We will first describe the process for $\mathbb{F}^2$, and then for $\mathbb{F}^3$

Process of Aligning Normals of $As^1(\star)$ and $Sq'^1(\star)$ in $\mathbb{F}^2$:
(I)Take $\triangle As^1(\star)$
(II) Fix points where $\triangle As^1(star)$ intersects $Ci^1(\star)$
(III) Reflect $\triangle As^1(p_i)$ over $\triangle As^1(p_{\frac{\pi}{4}})$
(Note: $p_{\frac{\pi}{4}}$ is the point that is equivalent to $\star = (r, \frac{\pi}{4})$, and $\angle(\triangle As^1(\star)), (\triangle Sq'^1(\star)) = 0$ here naturally, and at analogous points in QI, QII, $QIII$, and QIV).
(IV) End. Then $|Re(\triangle As^1(p_i)) - Ci^1(p_j)| = |As^1(p_j - Ci^1(p_j)$, which runs parallel to any $Sq'^1(p_j)$.

This explicitly states why [1] describes $max|As(p_i) - Ci(p_i)|$ as $max|As(p_{\frac{\pi}{4}}) - Ci(p_{\frac{\pi}{4}})|$. Note $|As(e_i) - Ci(e_i)| = 0$ for all points where $As^1(\star) \cap Ci^1(\star)$. In fact, $|As(e_i) - Ci(e_i)| = 0$ for all points where $As^n(\star) \cap Ci^n(\star)$ in all dimensions $n+1$. Thus, for $n = 1$, we only need to swap via reflection the open sets $(e_i, p_{\frac{\pi}{4}})$ and $(e_j, p_{\frac{\pi}{4}})$.
Note, via symmetry, that the transformation Re does not change the the overall structure of $gL(\star)$ or its component functions. For instance, both global and local curvature remain the same before and after the transformation Re is applied.

Process of Aligning Normals of $As^2(\star)$ and $Sq'^2(\star)$ in $\mathbb{F}^3$
(I)Take $\triangle As^2(\star)$
(II)Fix points where $\triangle As^2(\star)$ intersects $Ci^2(\star)$
(III) For boundary segments in the plane, follow Process of Aligning Normals of $As^1(\star)$ and $Sq'^1(\star)$ in $\mathbb{F}^2$
(IV) Rotate each open interior octant's triangulation of$\triangle As^2(\star)$ about centre of individual triangulation by $\frac{2\pi}{3}$.
(V) End. $|Ro(\triangle As^2(p_i)) - Ci^2(p_j)| = |As^2(p_j) - Ci^2(p_j)$, which runs parallel to any $\triangle Sq'^2(p_j)$.

This process untwists the intersection of normals under $\triangle As^2(\star)$ such that $\angle(\triangle As^2(\star)), (\triangle Sq'^2(\star)) = 0$.
As with Process of Aligning Normals of $As^1(\star)$ and $Sq'^1(\star)$ in $\mathbb{F}^2$, note that we needn't transform points associated with e_i, since normals are already aligned.

5 Conclusion

We've seen that we can align normals of $As^n(\star)$ and $Sq'^n(\star)$ for $\{n\} = \{1, 2\}$ in $n+1$ dimensions under given transformations. This (hopefully) addresses some of the ambiguity in previous papers dealing with development of a $gL(\star)$ space. For higher dimensions, the processes of untwisting/untangling the normal vectors is currently unknown to the author, but these processes (if they exist) are being looked into.
It's worth noting that the author, as of yet, can find no process in which $\angle(\triangle As^n(\star)), (\triangle Ci^n(\star)) = 0$ or $\angle(Sq'^n(\star)), (\triangle Ci^n(\star)) = 0$. This does not affect the construction of the $gL(\star)$ space, but the author does think such a process (if it can be shown to exist) would be interesting.

6 References

(1) Geometric Analytical Methods in Regards to Topology of 1-Manifolds of Constant Curvature (Spring 2015)
(2)Investigations into the Laplacians and Laplace Transforms with Respect to the Circle, the Aster, and Construction of an Analytic Geometric Lens (Winter 2015)
(3) Determinants and Applicable Methodology of Determinants with Respect to the Building and Understanding of Geometric Lens Theory (August 2016)
(4)Concentrations of Positive Curvature on Manifolds with Euler Characteristic 2 for Real Dimensions 3 and Higher (July 2017)
(5) Blanket Errata 1
(6) Blanket Errata 2
All references above are papers/erratas by Jonathan D. Evenboer.
All papers are self-published and can be found via links on the following web-page:
https://jonathandavidevenboer.weebly.com/cv.html
References given/listed in those papers and erratas are implied here.